Non integer orderer derivatives have been studied by mathematicians since the late seventeenth century. Such greats like Leibnitz, de L'Hospital, Riemman, Louville, Laplace and others were involved in their early development. However recent years showed a surge in interest caused by potential in applications. Non integer order differential equations were applied in such areas as

- superacpacitors,
- electrochemistry,
- anomalous diffusion,
- heat transfer through nonuniform media,
- stochastic processes in physics,
- and many others.

This also drawn the interest of researchers specialising in control. Podlubny [22] introduced an interesting concept such as $\mathrm{PI}^{\mu}\mathrm{D}^{\lambda}$ controller, which was a generalisation of a classical PID controller with non integer order integral and derivative.

Currently three definitions of non integer order derivative are being used. First of them, introduced by Riemann and Louville, introduces the derivative of order α

$$ {}_{RL}\mathrm{D}^{\alpha}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t}\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)\mathrm{d}\tau \tag{1} $$

where n is the nearest integer bigger than α and Γ function is given by

$$ \Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,\mathrm{d}t $$

Other definition, equivalent to the Riemann-Louville is the Grünwald-Letnikov definition

$$ {}_{GL}\mathrm{D}^{\alpha}x(t) = \lim_{\substack{h\to 0\\ h=t/m}} \frac{1}{h^{\alpha}}\sum_{k=0}^{m}(-1)^k\binom{\alpha}{k}x(t-kh) \tag{2} $$

where generalised Newton symbol is given by

$$ \binom{\alpha}{k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)}{j!} & \text{for} \quad j\in\mathbb{N} \\ 1 & \text{for} \quad j=0 \end{cases} \tag{3} $$

Final and most recently introduced definition (1967) is the definition by Caputo

$$ {}_{C}\mathrm{D}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau, \quad n=\lceil\alpha\rceil \tag{4} $$

Non integer order derivatives and non integer order differential equations (popularly known as fractional) have many interesting properties, which lead to their application and also problems with them. One of them is the fact that Laplace transform also functions for fractional operators, for example for zero initial conditions for all definitions we have

$$ \mathcal{L}\{\mathrm{D}^{\alpha}x(t)\} = s^{\alpha}X(s) \tag{5} $$

Differences are however for non zero initial conditions, as computation of Riemann-Louville and Grünwald-Letnikov derivatives (and differential equations) requires knowledge of initial values of fractional derivatives of a function and Caputo derivatives requires value of the function and its integer order derivatives.

Other important property of non integer order systems is the fact that they have infinite memory, that is the solution of non integer order differential equation

$$\mathrm{D}^{\alpha}x(t) = f(x(t)) \tag{6}$$

at given time instance t_1 requires the knowledge of the entire history from 0 to t_1. That is also the cause of implementation problems.

Non integer order systems in control engineering became especially interesting because of their frequency response properties. In the figure 1 one can see the frequency response of system

$$G_1(s) = \frac{1}{s^{\alpha}+1} \tag{7}$$

for different values of α.

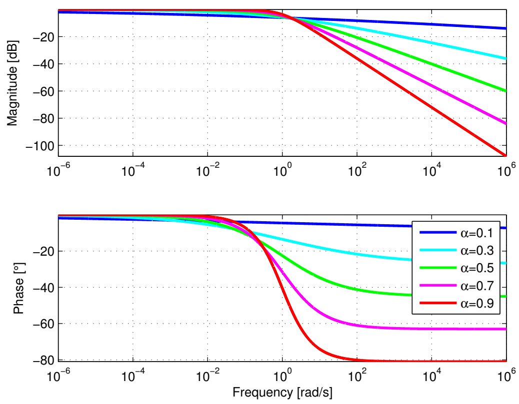

Figure 1: Frequency responses of system $G_1(s) = \frac{1}{s^{\alpha}+1}$

As it can be seen such simple system has very interesting responses, for example of low values of alpha even low frequencies are damped. Also it has important effects for controllers. The mentioned earlier $PI^{\mu}D^{\lambda}$ controller has also different characteristics. In the figure 2 one can see how integral and derivative elements have their frequency responses changed by different orders.

These attractive possibilities of new shapes of frequency characteristics sparked interest especially in field of robust control [18]. However the problem of implementation became significant. It is known how to solve differential equations of non integer order on any finite interval [10, 27, 30, 31] however controllers, filters and correctors in control systems need the possibility to operate on potentially infinite intervals. And because all history is needed because of the infinite memory it becomes a problem to find a solution for larger intervals. That led to a great amount of research on methods of approximation of non integer order systems [2–7, 11, 13, 15–17, 23, 24, 26]. One popular approach is to construct a difference approximation from Grünwald-Letnikov definition by replacing infinite sum with a finite one. This

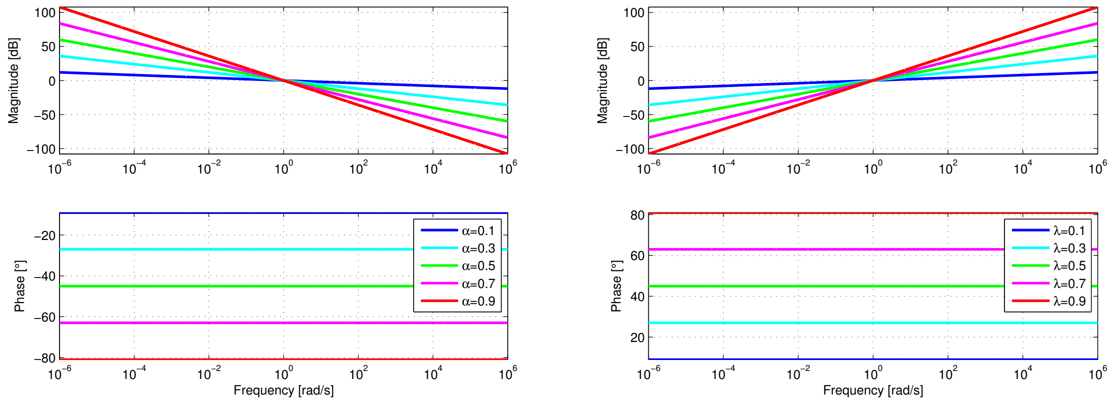

Figure 2: Frequency responses of integral and derivative parts of $PI^{\mu}D^{\lambda}$ controller

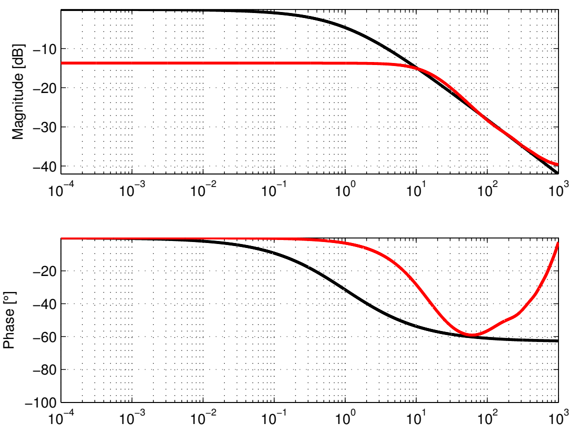

Figure 3: Frequency response of difference approximation of non integer order system

approximation however leads to substantially different characteristics, what can be observed in figure 3. As it can be observed they are consistent only for certain high frequencies. There are other kinds of approximations based on frequency domain fitting, which can lead to very consistent approximation via continuous system - these approximations are often of relatively high integer order (40 or more). However when attempting to discretise these approximation in order to obtain difference equations that can be implemented in digital systems destabilisation occurs. For certain methods it is caused only by coefficient quantisation on fixed point arithmetics [20,21], however different methods can even destabilise in floating point double precision arithmetic, one such example in figure 4 - poles of discretised system leave the unit circle.

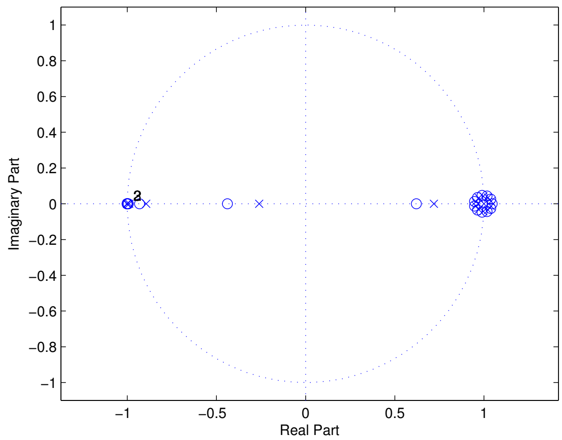

Figure 4: Poles of discretised approximation (computation on double precision)

Problem of approximations that can be properly discretised is relatively open in the context of fractional systems. Important results are available for integer order systems [8,28,29] however problems of discretisation and realisation are worth investigating. Similar lack of interest are for discretisations in the time domain, that preserve the frequency properties. This approach, that is natural in integer order is omitted in non integer order systems. Unpublished results of authors show very interesting behaviour that is worth investigating.

Also the problems of stability systems with non integer order controllers are less investigated. There are multiple results when both system and controller are linear [1, 9, 25] however application with nonlinear systems is almost untouched. There are methods similar to classical linearisation [19] but very little interest is given to Lyapunov like methods, which have great potential in applications [14].

Finally the problem of application of controllers on non integer order is left without systematic analysis. There are certain qualitative results with chaotic systems and general properties [12], there are however no systematic results regarding controller tuning. Moreover when attempting such tuning usually classical performance indexes are used, while not much focus was given what kind of optimisation criteria are best suited for this kind of systems.

The main reason for significance of this project is the aspect of transferring the more theoretical approach to the setting of control engineering. The well ground base of mathematical theory of non integer order systems allows analysis of their applications in real life setting.

Observation of only frequency characteristics of non integer order systems shows their great potential that is still not realised in the context of controller, filter and corrector design. Also other properties are very interesting, for example transitional behaviour can at certain periods be much faster than in integer order systems. In this project it is intended to verify the usefulness of different features of this class of systems and possibly propose new control paradigms.

Moreover stability of integer order systems with non integer order feedbacks is different very important problem, that had not found much coverage in literature. This project has an opportunity to obtain useful stability criteria based on Lyapunov-like methods and Mittag-Leffler stability concept. If such results would be obtained it would certainly be a significant step in understanding and applying fractional elements.

Also the experimental part of the project carries much value, as it would be one of the first systematic studies on using fractional controllers in highly nonlinear settings. Laboratory systems chosen for this research have very different and at the same time difficult behavioural features. If non integer order controllers or filters would improve their behaviour it would pave the way for very practical applications in future.

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